Technical Notes

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Rayleigh and Ritz Revisited

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THE use of the Rayleigh and Rayleigh-Ritz methods has long been a part of the stock in trade of aeronautical structural dynamicists and aeroelasticians, going back well before World War II. These methods were needed to represent complex aircraft structures by a manageable number of degrees of freedom or generalized coordinates in order to permit computation to be done with slide-rules and desk calculators. However, with the advent of the electronic digital computer, there came to the fore a new generation of specialists, whose backgrounds were not in dynamics and structures but rather in the newer fields of applied mathematics concerned with numerical analysis, computer software design, and programing. Members of the latter school of analysts, particularly in the early stages of their work in specific fields of application, have not concerned themselves with the economy of their computations. Rather they have harnessed the overwhelming computational power of modern digital computers to successfully carrying through "brute force" numerical solutions. In the case of structural dynamics and aeroelasticity, this has led to much less concern for reducing the number of degrees of freedom which must be handled in the solution of specific problems. Discrete element methods have lent themselves to the computer solution of many problems of structures and structural dynamics with large numbers of degrees of freedom (sometimes in the hundreds) and such computations have been used extensively in the analysis and design of large, high-performance aircraft and missiles.

Since the advent of the electronic digital computer era, a considerable body of literature on matrix numerical computation has grown up in which Rayleigh-Ritz methods are barely mentioned in passing. For example, the index to Householder's well-known book[1] contains no citation for either Rayleigh or Ritz. Some books (for example, Ref. 2) do refer to the Rayleigh quotient as a means of refining eigenvalues obtained by other means, including matrix iteration, but it is not discussed in the context of generalized coordinates or the reduction of the number of degrees of freedom to be considered, nor is Ritz's method usually treated.

In recent years, however, there has been a growing number of papers (many of them in AIAA publications) which, under various descriptive names, apply the concept of generalized coordinates to matrix numerical analyses, particularly of eigenvalue problems. For the most part, the problem considered has been the classical small-amplitude vibration problem, in which the eigenvalue is, to within multiplicative constants, the vibration frequency squared. The recent paper by Fried[3] demonstrates that a class of generalized coordinate methods in matrix analysis which have recently been designated as "condensation" techniques are really "a combination of the finite element method, the classical method of Rayleigh-Ritz and the power method." The particular approach of combining the one or more iterations (the power method) for the approximate eigenfunction (natural mode of vibration) with Rayleigh's Principle for determining the approximate lowest eigenvalue (natural frequency of vibration squared) is well-known and was set forth most clearly and completely by Temple and Bickley[4] for continuous systems. Moreover, even the details of the extension of this approach to discrete element numerical analyses are not new; for the matrix formulation of vibration problems, Duncan, Collar, and Frazer in their treatise[5] enunciated this approach and applied it to several examples. In theoretical physics, the name "variation-iteration methods" has been attached to the general class of techniques to which "condensation" belongs.[6]

The use of the power method in combination with the Rayleigh-Ritz method in approximations employing more than one degree of freedom to determine higher eigenvalues and eigenfunctions has been much less common. An instance of such use is to be found in Ref. 7 in an example of the application of graphical integration in the Rayleigh-Ritz method. A comprehensive treatment of various applications of the Rayleigh-Ritz method for multiple degrees of freedom to problems in matrix form has been given by Bisplinghoff, Ashley, and Halfman.[8] However, of the methods of approximating matrix eigenvalue problems by reducing the number of degrees of freedom considered, only one represents the application of the power method in combination with the Rayleigh-Ritz method employing more than a single assumed mode. This is Rauscher's method of station functions[9] which involves the construction of assumed modes which are special linear combinations of curves of deflection of the structure under loading. Because of the special character of these functions, the resulting condensed equations have the appearance of a collocation rather than a Rayleigh-Ritz formulation.

Thus it appears that some numerical analysts have come full circle: having started with techniques adapted to handling large numbers of degrees of freedom in matrix form without requiring resort to Rayleigh-Ritz methods, they have come back to Rayleigh and Ritz in seeking ways to reduce the number of degrees of freedom which must be handled. This Note is not an exposition of new techniques: rather it is intended to contribute to the educational process of melding the "new" numerical analysis methods into the older methods, which apparently have been overlooked by many analysts of the new school.

Rayleigh's method always leads to estimates for the lowest natural frequency, unless the mode shapes assumed are orthogonal (with mass as a weighting factor) to the mode of the lowest frequency. In the problem used as an example by Fried, because of the symmetry of discrete elements about the middle element and the fact that (for such systems) all symmetrical vibration modes are orthogonal to all antisymmetrical ones, Rayleigh's method can be used to obtain both the lowest symmetrical frequency and the lowest antisymmetrical frequency.

The Rayleigh-Ritz method is general in nature and is applicable whether a structural or vibration problem is expressed in the form of differential equations, integral equations, finite-difference equations, or matrices. The values of integrals and summations arising in the Rayleigh-Ritz method from expressions for the potential and kinetic energy of a dynamical system in terms of generalized coordinates may be calculated analytically, graphically, or numerically and different methods may be used for different terms if convenient or appropriate. For example, the terms arising from the potential energy may, in some cases, be calculated most conveniently analytically or graphically, while the terms arising from the kinetic energy may be more suitably handled by numerical summations.[10] Further, if discrete elements are used they may be different in identity and number for different terms, and various interpolation functions and

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quadrature schemes may be used in effecting the numerical integrations. In this sense, specific discrete element "condensation" techniques represent only one of many possible ways for carrying out the required process of integration and summation in the Rayleigh-Ritz method.

To further illustrate the alternative ways in which Rayleigh's principle may be applied and the fact that the use of a vibration mode determined from a first iteration does not necessarily lead to best results, the numerical example given by Fried[3] will be considered from the simpler (and more traditional) point of view of the dynamicist. The Rayleigh quotient may be formed from Fried's Eq. (6) by multiplying the first row of the matrix equation by x_1, the second by x_2, etc., and summing the results to give

$$\lambda = \frac{x_1^2+(x_2-x_1)^2+(x_3-x_2)^2+(x_4-x_3)^2+(x_5-x_4)^2+x_5^2}{x_1^2+x_2^2+x_3^2+x_4^2+x_5^2} \quad (1)$$

Apart from physical constants which are absorbed in the definition of λ, this may be viewed as Rayleigh's quotient for the transverse vibration of a massless string under constant tension with five equal masses equally spaced along the string. In this case the numerator represents the strain energy and the denominator represents the kinetic energy divided by frequency squared. In the traditional approach of the dynamicist, a reasonable (and typical) assumption for the shape of the lowest symmetric mode of this system would be a parabola (as in Appendix 1 of Ref. 4). Thus

$$x_1 = 5, \quad x_2 = 8, \quad x_3 = 9, \quad x_4 = 8, \quad x_5 = 5$$

Substituting these values into Eq. (1) gives

$$\lambda_1 = 70/259 = 0.2703$$

This compares to Fried's result for the singly iterated mode, $\lambda_1 = 0.2860$, for the doubly iterated mode, $\lambda_1 = 0.2680$, and the exact value, $\lambda_1 = 0.2679$. The accuracy of the result obtained here by application of the primitive Rayleigh method is thus much better than the result obtained by Fried using a singly iterated mode and differs from the exact solution by less than 1%.

For the antisymmetrical case, applying a similar assumed parabola to each half of the system leads to $\lambda_2 = 1.000$ which corresponds to the exact solution. By contrast, Fried obtained $\lambda_2 = 1.200$ for his singly iterated mode and $\lambda_2 = 1.024$ for his doubly iterated mode. Fried's results are a consequence of the fact that his assumed antisymmetric loading for the first iteration destroyed the subsymmetry of the first antisymmetric vibration mode in each half of the system.

This example is not, of course, typical of all problems to which Rayleigh's principle might be applied. (For instance, in the bending vibration of cantilever beams of variable properties use of the first iteration for the mode shape in application of Rayleigh's principle to determination of the lowest natural frequency is often a most satisfactory procedure.) It does, however, emphasize that to obtain best results with minimum effort from Rayleigh's principle, it is necessary for the analyst to exercise a degree of judgment based on the geometrical, structural, and dynamic properties of the system under consideration and to choose analytical techniques accordingly.

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Periodic Solutions of Gravity Oriented Axisymmetric Systems

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1. Introduction

THE periodic solutions of the planar librational motion for satellites in an eccentric orbit were first investigated by Zlatousov et al.[1] More recently, the analyses by Modi and associates[2–5] clearly established their usefulness in providing valuable information concerning the formation of a stability region and the critical eccentricity beyond which no stable motion is possible. Interestingly, at the critical eccentricity, the only available solution is a periodic one. Thus, periodic solutions play an important role in the attitude dynamics study.

This Note studies initial conditions leading to periodic motion for gravity gradient stabilized satellites using an extension of the Krylov and Bogoliubov method (variation of parameter procedure)[6] as suggested by Butenin[7] with certain modifications.[8,9] The validity of the results, obtained over a wide range of system parameters, is assessed through comparison with the response as given by numerical integration of the exact equations of motion. The effect of solar radiation pressure (the predominant environmental force at higher attitudes) is also included to emphasize its significance. The procedure represents a simple yet effective model for response evaluation during the preliminary design stage.

2. Equations of Motion and Analysis

Using the Lagrangian formulation, the governing equations for pitch (ψ), roll (β) and yaw (λ) librations (Fig. 1) for an arbitrarily shaped axisymmetric, gravity oriented satellite can be written as[10]

$$(1+e\cos\theta)\psi'' - 2e(1+\psi')\sin\theta - 2(1+e\cos\theta)(1+\psi')\beta'\tan\beta + 3K_i\sin\psi\cos\psi = Q_\psi \quad (1a)$$

$$(1+e\cos\theta)\beta'' - 2e\beta'\sin\theta + [(1+e\cos\theta)(1+\psi')^2 + 3K_i\cos^2\psi] * \sin\beta\cos\beta = Q_\beta \quad (1b)$$

$$\lambda' - (1+\psi')\sin\beta = 0 \quad (1c)$$

where e is the orbital eccentricity, θ is the position of the satellite as measured from pericenter, K_i is the inertia parameter, $(I_{xx}-I_{zz})/I_{yy}$, I_{jj} is the principal moments of inertia about j-axis ($j = x, y, z$), Q_ψ, Q_β are the generalized forces, complicated functions of geometry of the satellite, its orbit, mass distribution and spatial orientation, and primes denote differentiation with respect to θ.

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